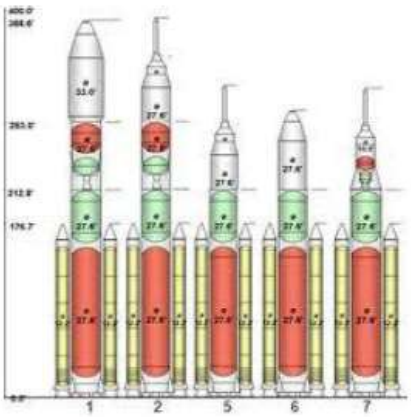
LASR
Land Air and Space Robotics

TEXAS A&M UNIVERSITY
AEROSPACE ENGINEERING

ATTITUDE ERROR KINEMATICS: APPLICATION IN OPTIMAL CONTROL OF DYNAMICAL SYSTEMS



AIAA Houston Annual Technical Symposium

17 May 2013

NASA/JSC, Gilruth Center, Houston, Tx.

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Outline

- **Introduction**
- **Optimal Reference Motion**
  - Open-loop solution
- **Motion Error Dynamics**
  - Angular Velocity Error: $\delta\omega = \omega - \omega_r$
  - Angular Velocity Error: $\delta\omega = \omega - \delta C\,\omega_r$
- **Generic Optimal Tracking Control Formulation**
  - Quadratic Penalty Function
  - Universal Penalty Function
  - MRP Attitude Penalty Function
- **Future Work and Conclusion**

ω_r projected onto the ***reference trajectory body axes***

ω_r projected onto the ***actual instantaneous body axes***

Introduction

- **Generic Optimal Control Solution --** *introduce a universal quadratic penalties of tracking errors that is consistent in each of the coordinate choices—i.e. a quadratic penalty on the MRPs error is clearly not "the same" physically as a quadratic penalty on the classical Rodrigues parameters*

  We utilize this universal attitude error measure expressed through approximate transformations as a positive function of each of the coordinate choices

>> Main Contribution

- ❑ Full nonlinear Motion Error in Compact Form
- ❑ Universe Penalty Function for O.C.P

Introduction

Example: (Spacecraft Maneuver)

$$\min J = \frac{1}{2}\int_{t_0}^{t_f}\left\{\boldsymbol{x}^T Q\boldsymbol{x} + \boldsymbol{u}^T R\boldsymbol{u}\right\}dt = \frac{1}{2}\int_{t_0}^{t_f}\left\{\boldsymbol{\omega}^T Q_{\omega}\boldsymbol{\omega} + \boldsymbol{\zeta}^T Q_{\zeta}\boldsymbol{\zeta} + \boldsymbol{u}^T R\boldsymbol{u}\right\}dt$$

$$\text{subject to: } \dot{\boldsymbol{x}} = \begin{cases} I\dot{\boldsymbol{\omega}} = -[\boldsymbol{\omega}\times]I\boldsymbol{\omega} + \boldsymbol{u} \\ \dot{\boldsymbol{\zeta}} = \left[\boldsymbol{f}(\boldsymbol{\zeta})\right]\boldsymbol{\omega} \end{cases}, \text{ where } Q = diag([Q_{\omega}, Q_{\zeta}])$$

where $\boldsymbol{\omega}$ is angular velocity and $\boldsymbol{\zeta}$ is an arbitrary attitude parameter

If Quaternion and the state weight was $Q = diag([Q_{\omega}, \alpha I_{4\times4}])$

where α *is a positive non – zero scalar*

$$J = \frac{1}{2}\int_{t_0}^{t_f}\left\{\alpha + \boldsymbol{\omega}^T Q_{\omega}\boldsymbol{\omega} + \boldsymbol{u}^T R\boldsymbol{u}\right\}dt$$

The cost function becomes independent of attitude variable due to quaternion norm constraint!!

Optimal Reference Motion

Minimize $J = \frac{1}{2}\Phi(t_f, \boldsymbol{\omega}(t_f), \boldsymbol{\zeta}(t_f)) + \frac{1}{2}\int_{t_0}^{t_f} L(\boldsymbol{\omega}, \boldsymbol{\zeta}, \boldsymbol{u}, t)dt$

Subject to $\dot{\boldsymbol{x}} = [\dot{\boldsymbol{\omega}}^T \quad \dot{\boldsymbol{\zeta}}^T]^T = \begin{cases} I\dot{\boldsymbol{\omega}} = -[\boldsymbol{\omega}\times]I\boldsymbol{\omega} + \boldsymbol{u} \\ \dot{\boldsymbol{\zeta}} = [\boldsymbol{f}(\boldsymbol{\zeta})]\boldsymbol{\omega} \end{cases}$

where the penalty functions are

$$\Phi(t_f, \boldsymbol{\zeta}(t_f), \boldsymbol{\omega}(t_f)) = Q_1 g(\boldsymbol{\zeta}_{t_f}) + \boldsymbol{\omega}_{t_f}^T Q_2 \boldsymbol{\omega}_{t_f}$$

and

$$L(\boldsymbol{\zeta}, \boldsymbol{\omega}, \boldsymbol{u}, t) = Q_3 g(\boldsymbol{\zeta}) + \boldsymbol{\omega}^T Q_4 \boldsymbol{\omega} + \boldsymbol{u}^T R \boldsymbol{u}$$

<u>*Hamiltonian*</u>

$$H = \frac{1}{2}\left\{Q_3 g(\boldsymbol{\zeta}) + \boldsymbol{\omega}^T Q_4 \boldsymbol{\omega} + \boldsymbol{u}^T R \boldsymbol{u}\right\} + \boldsymbol{\lambda}_{\omega}^T I^{-1}\left(-[\boldsymbol{\omega}\times]I\boldsymbol{\omega} + \boldsymbol{u}\right) + \boldsymbol{\lambda}_{\zeta}^T\left[\boldsymbol{f}(\boldsymbol{\zeta})\right]\boldsymbol{\omega}$$

Optimal Reference Motion

Invoking the Pontryagin necessary condition for optimality:

$$\dot{\boldsymbol{\lambda}}_{\omega} = -Q_4\boldsymbol{\omega} - \left[\boldsymbol{f}(\boldsymbol{\zeta})\right]^T \boldsymbol{\lambda}_{\zeta} - \left(I[\boldsymbol{\omega}\times] - [(I\boldsymbol{\omega})\times]\right) I^{-1}\boldsymbol{\lambda}_{\omega}$$

$$\dot{\boldsymbol{\lambda}}_{\zeta} = -\frac{1}{2}Q_3\frac{\partial g}{\partial \boldsymbol{\zeta}} - \frac{\partial}{\partial \boldsymbol{\zeta}}\left[\left[\boldsymbol{f}(\boldsymbol{\zeta})\right]\boldsymbol{\omega}\right]^T \boldsymbol{\lambda}_{\zeta}$$

$$\boldsymbol{u} = -R^{-1}I^{-1}\boldsymbol{\lambda}_{\omega} = -(IR)^{-1}\boldsymbol{\lambda}_{\omega}$$

What are $g(\,)$ & $\boldsymbol{f}(\,)$??

Table 1: Attitude Kinematics.

$\boldsymbol{\zeta}$ $\qquad$ $\dot{\boldsymbol{\zeta}} = [\boldsymbol{f}(\boldsymbol{\zeta})]\boldsymbol{\omega}$

Attitude parameter	Attitude Kinematics
DCM	$\dot{C} = -[\boldsymbol{\omega}\times]C$
Quaternion	$\dot{\boldsymbol{q}} = \frac{1}{2}\begin{bmatrix} -[\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix}\begin{Bmatrix} \boldsymbol{q}_v \\ q_4 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$
CRPs	$\dot{\boldsymbol{\rho}} = \frac{1}{2}\left[[I_{3\times3}] + [\boldsymbol{\rho}\times] + \boldsymbol{\rho}\boldsymbol{\rho}^T\right]\boldsymbol{\omega}$
MRPs	$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[(1-\boldsymbol{\sigma}^T\boldsymbol{\sigma})[I_{3\times3}] + 2[\boldsymbol{\sigma}\times] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\boldsymbol{\omega}$
Euler Angles*	$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}_{ijk} = \mathbf{S}^{-1}(\theta_1, \theta_2, \theta_3)\boldsymbol{\omega}$
Principal angle/axis	$\dot{\phi} = \boldsymbol{\omega}^T\hat{\boldsymbol{e}}, \quad \dot{\boldsymbol{e}} = \frac{1}{2}\left[[\hat{\boldsymbol{e}}\times] - \cot(\phi/2)[\hat{\boldsymbol{e}}\times][\hat{\boldsymbol{e}}\times]\right]\boldsymbol{\omega}$
Cayley-Klein	$\mathrm{col}(\dot{K}) = \frac{1}{2}\Psi_0\begin{bmatrix} -[\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix}\Psi_0^{-1}\mathrm{col}(K), \quad \text{where } \Psi_0 = \begin{bmatrix} 0 & 0 & i & 1 \\ i & -1 & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & -i & 1 \end{bmatrix}$

*See Ref.[2] for $\mathbf{S}^{-1}(\delta\theta_1, \delta\theta_2, \delta\theta_3)$ definition.

Optimal Reference Motion

The scalar function $g(\ )$ is a general nonnegative attitude penalty function. The function is chosen that will produce identical performance index values, regardless of the attitude variables selected[1]

$$g\left(\left[C(\hat{\boldsymbol{e}},\phi)\right]\right)=\frac{1}{4}\left\{3-\mathrm{trace}\left(\left[C\right]\right)\right\}=\sin^2(\phi/2)$$

Quaternion

$$g\left(\left[C(\boldsymbol{q})\right]\right)=q_1^2+q_2^2+q_3^2 \quad and \quad \frac{\partial g}{\partial \boldsymbol{q}}=2\begin{bmatrix}q_1 & q_2 & q_3 & 0\end{bmatrix}^T$$

CRP

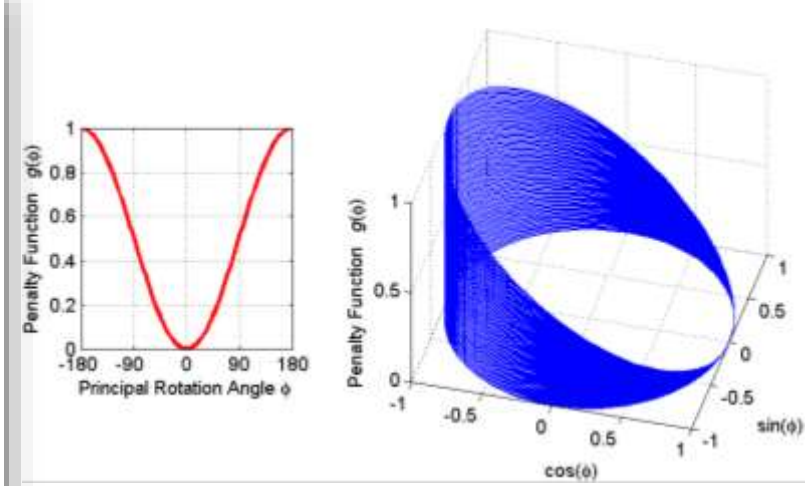
$$g\left(\left[C(\boldsymbol{\rho})\right]\right)=\frac{\boldsymbol{\rho}^T\boldsymbol{\rho}}{1+\boldsymbol{\rho}^T\boldsymbol{\rho}} \quad and \quad \frac{\partial g}{\partial \boldsymbol{\rho}}=\frac{2\boldsymbol{\rho}}{\left(1+\boldsymbol{\rho}^T\boldsymbol{\rho}\right)^2}$$

MRP

$$g\left(\left[C(\boldsymbol{\sigma})\right]\right)=4\frac{\boldsymbol{\sigma}^T\boldsymbol{\sigma}}{\left(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)^2} \quad and \quad \frac{\partial g}{\partial \boldsymbol{\sigma}}=8\boldsymbol{\sigma}\frac{1-\boldsymbol{\sigma}^T\boldsymbol{\sigma}}{\left(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)^3}$$

Euler Angles

$$g\left(\left[C(\boldsymbol{\theta})\right]\right)=\frac{1}{4}\left[3-(1+\cos(\theta_2))\cos(\theta_1+\theta_3)-\cos(\theta_2)\right] \quad and \quad \frac{\partial g}{\partial \boldsymbol{\theta}}=\frac{1}{4}\begin{bmatrix}(1+\cos(\theta_2))\sin(\theta_1+\theta_3)\\ \sin(\theta_2)\cos(\theta_1+\theta_3)+\sin(\theta_2)\\ (1+\cos(\theta_2))\sin(\theta_1+\theta_3)\end{bmatrix}$$



$$0 \le g(C) \le 1$$

(1) H. Schaub, J. L. Junkins, and R. D. Robinett, "*New Penalty Functions and Optimal Control Formulation for Spacecraft Attitude Control Problems*," Journal of Guidance, Control, and Dynamics, 20(3):428-434, May-June 1997.

Optimal Reference Motion

$Q_\omega = Q_\zeta = R = I_{3\times3}$

Open-Loop Optimal Control Problem

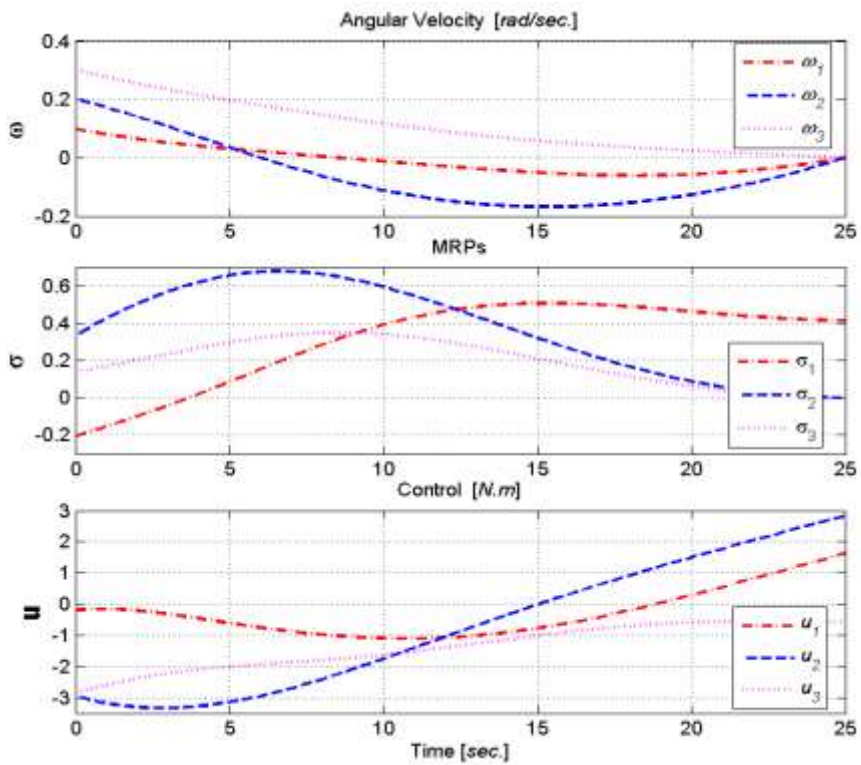
$\boldsymbol{\omega}(0) = [0.1, 0.2, 0.3]$, $\boldsymbol{\omega}(25) = [0, 0, 0]$ rad/sec

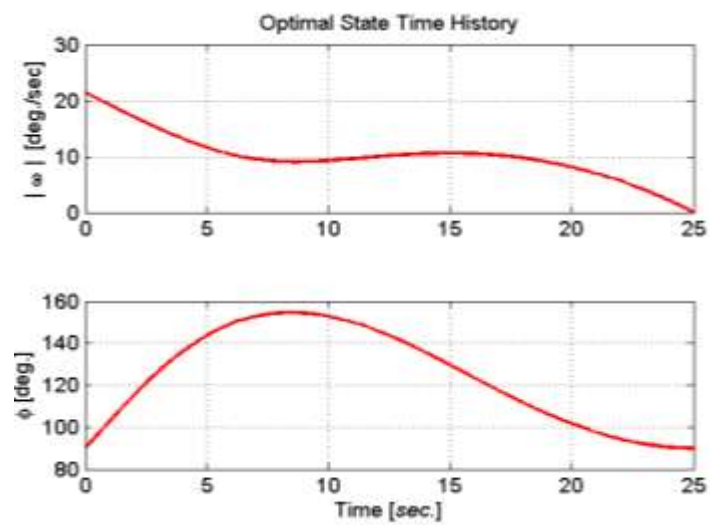
$\boldsymbol{\zeta}(0) = [-0.20295, 0.33388, 0.13750]$

$\boldsymbol{\zeta}(25) = [0.41421, 0, 0]$

$Q_1 = 0,\ Q_3 = 1,\ Q_2 = 0_{3\times3},$

$Q_4 = R = I_{3\times3}$ (Identity)





$I = diag([86.215, 85.070, 113.565])\quad Kg.m^2$

Angular Velocity Error Dynamics (Tracking Control Formulation)

The desired motion is defined in terms of the open-loop reference angular velocity

$$I_r \dot{\boldsymbol{\omega}}_r = -[\boldsymbol{\omega}_r \times] I_r \boldsymbol{\omega}_r + \tau$$

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$$

Angular Velocity Error Vector

$$\boxed{\delta\dot{\boldsymbol{\omega}} = -I^{-1}\left\{[\boldsymbol{\omega}_r \times] I - [(I\boldsymbol{\omega}_r)\times]\right\}\delta\boldsymbol{\omega} - I^{-1}[\delta\boldsymbol{\omega}\times] I \delta\boldsymbol{\omega} + I^{-1}\boldsymbol{u} - \dot{\boldsymbol{\omega}}_r - I^{-1}[\boldsymbol{\omega}_r \times] I \boldsymbol{\omega}_r}$$

Full Nonlinear Angular Velocity Error Dynamics Rate

For An Exact Kinematic Model This Equation Predicts Arbitrarily Large Motion Angular Velocity Error Vector Estimates (no linearization)

Motion Error Kinematics

Quaternion Error

$$\delta \boldsymbol{q} = \boldsymbol{q} \otimes \boldsymbol{q}_r^{-1} \Rightarrow \delta \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}} \otimes \boldsymbol{q}_r^{-1} + \boldsymbol{q} \otimes \dot{\boldsymbol{q}}_r^{-1}$$

$$\dot{\boldsymbol{q}} = \frac{1}{2}\begin{Bmatrix} \boldsymbol{\omega} \\ 0 \end{Bmatrix} \otimes \boldsymbol{q} = \begin{bmatrix} -[\boldsymbol{\omega}\times] & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega} & 0 \end{bmatrix} \boldsymbol{q} = \frac{1}{2}\Omega(\boldsymbol{\omega})\boldsymbol{q}$$

$$\dot{\boldsymbol{q}}_r^{-1} = -\frac{1}{2}\boldsymbol{q}_r^{-1} \otimes \begin{Bmatrix} \boldsymbol{\omega}_r \\ 0 \end{Bmatrix} = -\frac{1}{2}\Gamma(\boldsymbol{\omega}_r)\boldsymbol{q}_r^{-1}$$

$$\boxed{\delta \dot{\boldsymbol{q}} = \frac{1}{2}\left[\Omega(\boldsymbol{\omega}) - \Gamma(\boldsymbol{\omega}_r)\right]\delta \boldsymbol{q}} \Rightarrow \boxed{\delta \dot{\boldsymbol{q}} = \frac{1}{2}\left[\Omega(\delta\boldsymbol{\omega} + \boldsymbol{\omega}_r) - \Gamma(\boldsymbol{\omega}_r)\right]\delta \boldsymbol{q} = \frac{1}{2}\left[\Omega(\delta\boldsymbol{\omega}) + \Gamma(\boldsymbol{\omega}_r)\right]\delta \boldsymbol{q}}$$

Key Step

$$\boxed{\delta \dot{\boldsymbol{q}} = \frac{1}{2}\begin{bmatrix} -\left([\delta\boldsymbol{\omega}\times] + 2[\boldsymbol{\omega}_r\times]\right) & \delta\boldsymbol{\omega} \\ \delta\boldsymbol{\omega}^T & 0 \end{bmatrix}\begin{Bmatrix} \delta \boldsymbol{q}_v \\ \delta q_4 \end{Bmatrix}}$$

Rate Valid for Large Changes

General Procedure to Build Large Motion Error Models

Build on Nonlinear Plant & Quaternion Error Models

$$\delta \boldsymbol{q} = \boldsymbol{q} \otimes \hat{\boldsymbol{q}}^{-1} \qquad \delta \dot{\boldsymbol{q}} = \boldsymbol{f}\left(\delta \boldsymbol{q}, \delta \boldsymbol{\omega}\right)$$

Exact Error Models

Build New Variable Mapping Eqns

$$\delta \boldsymbol{\xi} = \boldsymbol{g}\left(\delta \boldsymbol{q}\right)$$

Exact Error Mapping

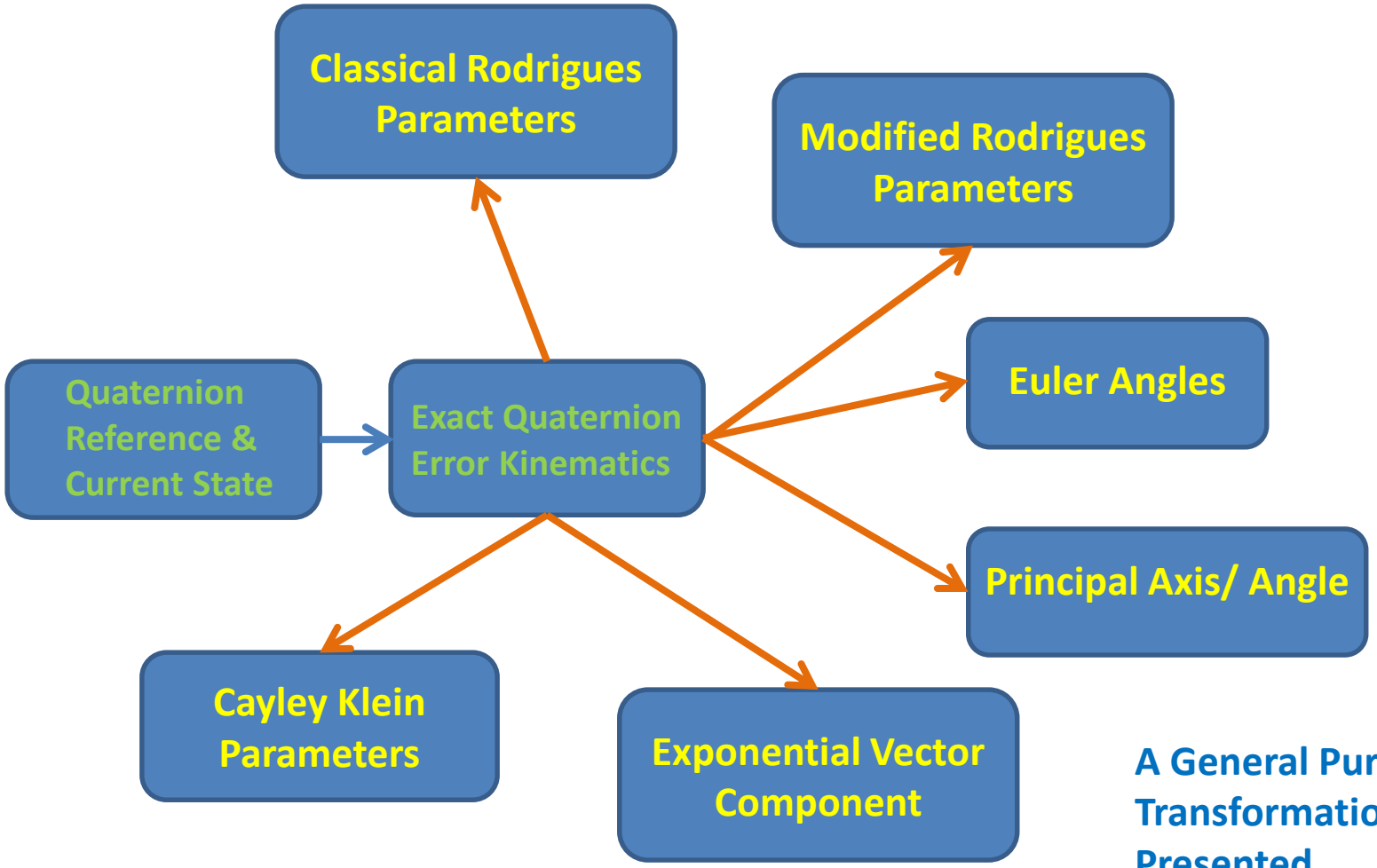
$$\delta \dot{\boldsymbol{\xi}} = \boldsymbol{h}\left(\delta \boldsymbol{q}, \delta \dot{\boldsymbol{q}}\right)$$

$$\boxed{\delta \dot{\boldsymbol{\xi}} = \boldsymbol{\phi}\left(\delta \boldsymbol{\xi}, \delta \boldsymbol{\omega}\right)}$$

Exact *Kinematics* Error Models

KEY INSIGHT: Exploit Exact Quaternion Error Model in Transformation of New Variable Sets. Leads to Models without Truncation Error for Large & Rapid Motions

Nonlinear Model Development



A General Purpose Transformation is Presented

Motion Error Kinematics

Attitude Error	$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$
DCM	$\delta\dot{C} = [\delta\boldsymbol{\omega}\times]\delta C - [\boldsymbol{\omega}_r\times]\delta C + \delta C[\boldsymbol{\omega}_r\times]$
Quaternion	$\delta\dot{\boldsymbol{q}} = \frac{1}{2}\begin{bmatrix} -([\delta\boldsymbol{\omega}\times] + 2[\boldsymbol{\omega}_r\times]) & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^T & 0 \end{bmatrix}\begin{Bmatrix} \delta\boldsymbol{q}_v \\ \delta q_4 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} \delta q_4 & -\delta q_3 & \delta q_2 \\ \delta q_3 & \delta q_4 & -\delta q_1 \\ -\delta q_2 & \delta q_1 & \delta q_4 \\ -\delta q_1 & -\delta q_2 & -\delta q_3 \end{bmatrix}\begin{bmatrix} \delta\omega_1 \\ \delta\omega_2 \\ \delta\omega_3 \end{bmatrix} - \begin{Bmatrix} [\boldsymbol{\omega}_r\times]\delta\boldsymbol{q}_v \\ 0 \end{Bmatrix}$
CRPs	$\delta\dot{\boldsymbol{\rho}} = \frac{1}{2}\left[[I_{3\times3}] + [\delta\boldsymbol{\rho}\times] + \delta\boldsymbol{\rho}\delta\boldsymbol{\rho}^T\right]\delta\boldsymbol{\omega} - [\boldsymbol{\omega}_r\times]\delta\boldsymbol{\rho}$
MRPs	$\delta\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[(1 - \delta\boldsymbol{\sigma}^T\delta\boldsymbol{\sigma})[I_{3\times3}] + 2[\delta\boldsymbol{\sigma}\times] + 2\delta\boldsymbol{\sigma}\delta\boldsymbol{\sigma}^T\right]\delta\boldsymbol{\omega} - [\boldsymbol{\omega}_r\times]\delta\boldsymbol{\sigma}$
Euler Angles*	$\begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \\ \delta\dot{\theta}_3 \end{bmatrix}_{ijk} = \frac{1}{4}(H_{ijk}^T H_{ijk}^T)^{-1} H_{ijk}^T \begin{bmatrix} -([\delta\boldsymbol{\omega}\times] + 2[\boldsymbol{\omega}_r\times]) & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^T & 0 \end{bmatrix}\Theta_{ijk}$
Principal angle/axis	$\delta\dot{\phi} = \delta\boldsymbol{\omega}^T\delta\hat{\boldsymbol{e}} = \delta\hat{\boldsymbol{e}}^T\delta\boldsymbol{\omega}$ $\delta\dot{\boldsymbol{e}} = \frac{1}{2}\left[[\delta\hat{\boldsymbol{e}}\times] - \cot(\delta\phi/2)[\delta\hat{\boldsymbol{e}}\times][\delta\hat{\boldsymbol{e}}\times]\right]\delta\boldsymbol{\omega} - [\boldsymbol{\omega}_r\times]\delta\hat{\boldsymbol{e}}$
Cayley-Klein	$\mathrm{col}(\delta\dot{K}) = \frac{1}{2}\Psi_0\begin{bmatrix} -([\delta\boldsymbol{\omega}\times] + 2[\boldsymbol{\omega}_r\times]) & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^T & 0 \end{bmatrix}\Psi_0^{-1}\mathrm{col}(\delta K)$ where $\Psi_0 = \begin{bmatrix} 0 & 0 & i & 1 \\ i & -1 & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & -i & 1 \end{bmatrix}$

*See Ref.[1] for H_{ijk} and Θ_{ijk} definitions, and see Ref.[2] for $\mathbf{S}^{-1}(\delta\theta_1, \delta\theta_2, \delta\theta_3)$ definition.

[1] Bani Younes, A., Mortari, D., Turner, J.D., and Junkins, J.L. "Attitude Error Kinematics," AIAA Journal of Guidance, Control, and Dynamics, accepted.
[2] P.C. Hughes. Spacecraft attitude dynamics. J. Wiley, 1986.

Motion Error Dynamics

Angular Velocity Error Vector $\delta\omega = \omega - \delta C\,\omega_r$

$$\delta\dot{\omega} = -I^{-1}\{[(\delta C\omega_r)\times]I - [(I\delta C\omega_r)\times]\}\delta\omega - I^{-1}[\delta\omega\times]I\delta\omega + I^{-1}u + [\delta\omega\times]\delta C\omega_r - \dot{\omega}_r - I^{-1}[(\delta C\omega_r)\times]I\delta C\omega_r$$

Attitude Error Vector

$\delta C = C\hat{C}^T$ note: $\dot{C} = -[\omega\times]C$ and $\dot{\hat{C}} = -[\omega_r\times]\hat{C}$

$\Rightarrow \delta\dot{C} = \dot{C}\hat{C}^T + C\dot{\hat{C}}^T = -[\omega\times]\delta C + \delta C[\omega_r\times]$ use $\delta\omega = \omega - \delta C\omega_r$

$\Rightarrow \delta\dot{C} = -[\delta\omega\times]\delta C - [(\delta C\omega_r)\times]\delta C + \delta C[\omega_r\times]$

transformation of skew-symmetric tensor identity $[(\delta C\omega_r)\times] = \delta C[\omega_r\times]\delta C^T$

$$\delta\dot{C} = -[\delta\omega\times]\delta C$$

attitude error kinematics equation is similar to the attitude kinematics equation!!

Motion Error Kinematics

Table 2: Attitude Error Kinematics.

Attitude Error	$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$	$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \delta C\boldsymbol{\omega}_r$
DCM	$\delta\dot{C} = [\delta\boldsymbol{\omega}\times]\delta C - [\boldsymbol{\omega}_r\times]\delta C + \delta C[\boldsymbol{\omega}_r\times]$	$\delta\dot{C} = -[\delta\boldsymbol{\omega}\times]\,\delta C$
Quaternion	$\delta\dot{\boldsymbol{q}} = \frac{1}{2}\begin{bmatrix} -([\delta\boldsymbol{\omega}\times] + 2[\boldsymbol{\omega}_r\times]) & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^T & 0 \end{bmatrix}\begin{Bmatrix} \delta\boldsymbol{q}_v \\ \delta q_4 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} \delta q_4 & -\delta q_3 & \delta q_2 \\ \delta q_3 & \delta q_4 & -\delta q_1 \\ -\delta q_2 & \delta q_1 & \delta q_4 \\ -\delta q_1 & -\delta q_2 & -\delta q_3 \end{bmatrix}\begin{bmatrix} \delta\omega_1 \\ \delta\omega_2 \\ \delta\omega_3 \end{bmatrix} - \begin{Bmatrix} [\boldsymbol{\omega}_r\times]\delta\boldsymbol{q}_v \\ 0 \end{Bmatrix}$	$\delta\dot{\boldsymbol{q}} = \frac{1}{2}\begin{bmatrix} -[\delta\boldsymbol{\omega}\times] & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^T & 0 \end{bmatrix}\begin{Bmatrix} \delta\boldsymbol{q}_v \\ \delta q_4 \end{Bmatrix}$
CRPs	$\delta\dot{\boldsymbol{\rho}} = \frac{1}{2}\left[[I_{3\times3}] + [\delta\boldsymbol{\rho}\times] + \delta\boldsymbol{\rho}\delta\boldsymbol{\rho}^T\right]\delta\boldsymbol{\omega} - [\boldsymbol{\omega}_r\times]\delta\boldsymbol{\rho}$	$\delta\dot{\boldsymbol{\rho}} = \frac{1}{2}\left[[I_{3\times3}] + [\delta\boldsymbol{\rho}\times] + \delta\boldsymbol{\rho}\delta\boldsymbol{\rho}^T\right]\delta\boldsymbol{\omega}$
MRPs	$\delta\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[(1 - \delta\boldsymbol{\sigma}^T\delta\boldsymbol{\sigma})[I_{3\times3}] + 2[\delta\boldsymbol{\sigma}\times] + 2\delta\boldsymbol{\sigma}\delta\boldsymbol{\sigma}^T\right]\delta\boldsymbol{\omega} - [\boldsymbol{\omega}_r\times]\delta\boldsymbol{\sigma}$	$\delta\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[(1 - \delta\boldsymbol{\sigma}^T\delta\boldsymbol{\sigma})[I_{3\times3}] + 2[\delta\boldsymbol{\sigma}\times] + 2\delta\boldsymbol{\sigma}\delta\boldsymbol{\sigma}^T\right]\delta\boldsymbol{\omega}$
Euler Angles*	$\begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \\ \delta\dot{\theta}_3 \end{bmatrix}_{ijk} = \frac{1}{4}(H_{ijk}^T H_{ijk}^T)^{-1} H_{ijk}^T \begin{bmatrix} -([\delta\boldsymbol{\omega}\times] + 2[\boldsymbol{\omega}_r\times]) & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^T & 0 \end{bmatrix}\Theta_{ijk}$	$\begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \\ \delta\dot{\theta}_3 \end{bmatrix}_{ijk} = \mathbf{S}^{-1}(\delta\theta_1, \delta\theta_2, \delta\theta_3)\delta\boldsymbol{\omega}$
Principal angle/axis	$\delta\dot{\phi} = \delta\boldsymbol{\omega}^T\delta\hat{\boldsymbol{e}} = \delta\hat{\boldsymbol{e}}^T\delta\boldsymbol{\omega}$ $\delta\dot{\boldsymbol{e}} = \frac{1}{2}\left[[\delta\hat{\boldsymbol{e}}\times] - \cot(\delta\phi/2)[\delta\hat{\boldsymbol{e}}\times][\delta\hat{\boldsymbol{e}}\times]\right]\delta\boldsymbol{\omega} - [\boldsymbol{\omega}_r\times]\delta\hat{\boldsymbol{e}}$	$\delta\dot{\phi} = \delta\boldsymbol{\omega}^T\delta\hat{\boldsymbol{e}} = \delta\hat{\boldsymbol{e}}^T\delta\boldsymbol{\omega}$ $\delta\dot{\boldsymbol{e}} = \frac{1}{2}\left[[\delta\hat{\boldsymbol{e}}\times] - \cot(\delta\phi/2)[\delta\hat{\boldsymbol{e}}\times][\delta\hat{\boldsymbol{e}}\times]\right]\delta\boldsymbol{\omega}$
Cayley-Klein	$\mathrm{col}(\delta\dot{K}) = \frac{1}{2}\Psi_0\begin{bmatrix} -([\delta\boldsymbol{\omega}\times] + 2[\boldsymbol{\omega}_r\times]) & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^T & 0 \end{bmatrix}\Psi_0^{-1}\mathrm{col}(\delta K)$ where $\Psi_0 = \begin{bmatrix} 0 & 0 & i & 1 \\ i & -1 & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & -i & 1 \end{bmatrix}$	$\mathrm{col}(\delta\dot{K}) = \frac{1}{2}\Psi_0\begin{bmatrix} -[\delta\boldsymbol{\omega}\times] & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^T & 0 \end{bmatrix}\Psi_0^{-1}\mathrm{col}(\delta K)$ where $\Psi_0 = \begin{bmatrix} 0 & 0 & i & 1 \\ i & -1 & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & -i & 1 \end{bmatrix}$

*See Ref.[1] for H_{ijk} and Θ_{ijk} definitions, and see Ref.[2] for $\mathbf{S}^{-1}(\delta\theta_1, \delta\theta_2, \delta\theta_3)$ definition.

[1] Bani Younes, A., Mortari, D., Turner, J.D., and Junkins, J.L. "Attitude Error Kinematics," AIAA Journal of Guidance, Control, and Dynamics, accepted.
[2] P.C. Hughes. Spacecraft attitude dynamics. J. Wiley, 1986.

Optimal Tracking Control

Quadratic Penalty Function

Minimize $$J = \frac{1}{2}\Phi(t_f, x_i(t_f)) + \frac{1}{2}\int_{t_0}^{t_f} \left\{ q_{j_3 j_4} x_{j_3} x_{j_4} + r_{j_5 j_6} u_{j_5} u_{j_6} \right\} dt$$

Subject to $$\dot{x}_i = a_{ij_1} x_{j_1} + c_{ijkl} x_j x_k x_l + t_{imn} x_m x_n + d_i + b_{ij_2} u_{j_2}$$

Co-states $$\dot{\lambda}_i = -q_{ij_4} x_{j_4} - \lambda_{i_1} \left\{ a_{i_1 i} + c_{i_1 jki} x_j x_k + c_{i_1 jil} x_j x_l + c_{i_1 ikl} x_k x_l + t_{i_1 mi} x_m + t_{i_1 in} x_n \right\}$$

Co-states (assumed form) $$\lambda_{i_4} = k_{i_4 j_8} x_{j_8} + p_{i_4}$$

Control Gains:

$$\dot{p}_i = -k_{ij} d_j + k_{ij} b_{jk} r_{lk}^{-1} b_{ml} p_m - p_n a_{ni}, \quad p_i(t_f) = 0$$

$$\dot{k}_{ij} = -k_{kj} a_{ki} + k_{il} b_{lm} r_{nm}^{-1} b_{on} k_{oj} - k_{il} a_{lj} - q_{ij} - p_r t_{rji} - p_r t_{rij}, \quad k_{ij}(t_f) = \mathrm{K}_f$$

Optimal Tracking Control

$Q_\omega = Q_\zeta = R = I_{3\times3}$

Closed-Loop Optimal Control Problem

$$\boldsymbol{\omega}(0) = [-0.1, -0.2, -0.3]\ \text{rad/sec},\ \boldsymbol{\zeta}(0) = [-2.4142, 0, 0]$$

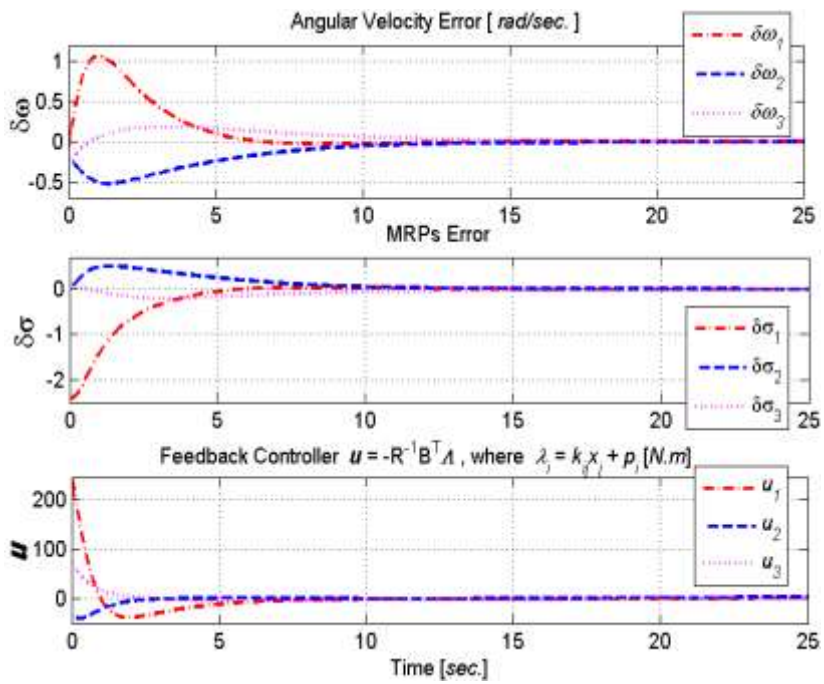
$$I = diag([86.215, 85.070, 113.565])\quad Kg.m^2$$

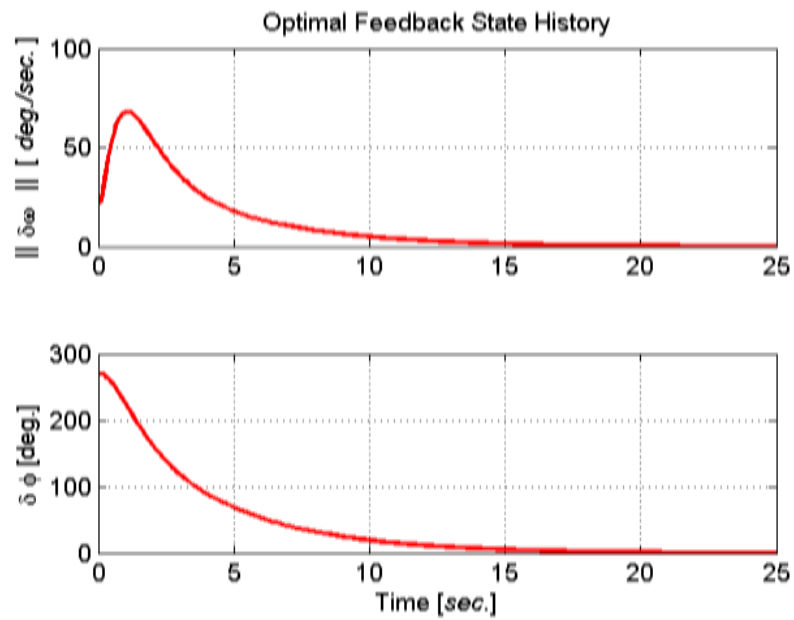
$$Q_1 = 0,\ Q_3 = 1,\ Q_2 = 0_{3\times3},$$

$$Q_4 = I_{3\times3}\ ,\ R = 10^{-4} I_{3\times3}$$

$$p_i = 0 \ \text{ and } \ k_{ij} = \delta_{ij}\ 10^3$$

$$\text{where} \begin{cases} \delta_{ij} = 0 & \text{if } i \neq j \\ \delta_{ij} = 1 & \text{if } i = j \end{cases}$$





Optimal Tracking Control

Universal Penalty Function

Minimize

$$J = \frac{1}{2}\left\{Q_1 g(\delta\zeta_{t_f}) + \delta\boldsymbol{\omega}_{t_f}^T Q_2 \delta\boldsymbol{\omega}_{t_f}\right\} + \frac{1}{2}\int_{t_0}^{t_f}\left\{Q_3 g(\delta\zeta) + \delta\boldsymbol{\omega}^T Q_4 \delta\boldsymbol{\omega} + \boldsymbol{u}^T R\boldsymbol{u}\right\}dt$$

Subject to

$$\dot{\boldsymbol{x}} = [\delta\dot{\boldsymbol{\omega}}^T \quad \delta\dot{\boldsymbol{\zeta}}^T]^T = \boldsymbol{f}(\delta\boldsymbol{\omega}, \delta\boldsymbol{\zeta}, \boldsymbol{u})$$

Universe Penalty Function

$$g\left(\left[\delta C(\delta\hat{\boldsymbol{e}}, \delta\phi)\right]\right) = \frac{1}{4}\left\{3 - \mathrm{trace}\left([\delta C]\right)\right\} = \sin^2(\delta\phi/2)$$

Quaternion

$$g\left(\left[\delta C(\delta\boldsymbol{q})\right]\right) = \delta q_1^2 + \delta q_2^2 + \delta q_3^2 \quad and \quad \frac{\partial g}{\partial(\delta\boldsymbol{q})} = 2[\delta q_1 \quad \delta q_2 \quad \delta q_3 \quad 0]^T$$

CRP

$$g\left(\left[\delta C(\delta\boldsymbol{\rho})\right]\right) = \frac{\delta\boldsymbol{\rho}^T\delta\boldsymbol{\rho}}{1+\delta\boldsymbol{\rho}^T\delta\boldsymbol{\rho}} \quad and \quad \frac{\partial g}{\partial(\delta\boldsymbol{\rho})} = \frac{2\delta\boldsymbol{\rho}}{\left(1+\delta\boldsymbol{\rho}^T\delta\boldsymbol{\rho}\right)^2}$$

MRP

$$g\left(\left[\delta C(\delta\boldsymbol{\sigma})\right]\right) = 4\frac{\delta\boldsymbol{\sigma}^T\delta\boldsymbol{\sigma}}{\left(1+\delta\boldsymbol{\sigma}^T\delta\boldsymbol{\sigma}\right)^2} \quad and \quad \frac{\partial g}{\partial(\delta\boldsymbol{\sigma})} = 8\delta\boldsymbol{\sigma}\frac{1-\delta\boldsymbol{\sigma}^T\delta\boldsymbol{\sigma}}{\left(1+\delta\boldsymbol{\sigma}^T\delta\boldsymbol{\sigma}\right)^3}$$

Optimal Tracking Control

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$$

$$\delta\dot{\boldsymbol{\omega}} = -I^{-1}\left\{[\boldsymbol{\omega}_r\times]I - [(I\boldsymbol{\omega}_r)\times]\right\}\delta\boldsymbol{\omega} - I^{-1}[\delta\boldsymbol{\omega}\times]I\delta\boldsymbol{\omega} + I^{-1}\boldsymbol{u} - \dot{\boldsymbol{\omega}}_r - I^{-1}[\boldsymbol{\omega}_r\times]I\boldsymbol{\omega}_r$$

$$\delta\dot{\boldsymbol{\zeta}} = f(\delta\boldsymbol{\zeta})\delta\boldsymbol{\omega} - [\boldsymbol{\omega}_r\times]\delta\boldsymbol{\zeta}$$

$$\dot{\boldsymbol{\lambda}}_{\delta\omega} = -Q_4\delta\boldsymbol{\omega} - f\left(\delta\boldsymbol{\zeta}\right)^T\boldsymbol{\lambda}_{\delta\zeta} - \left(I[\delta\boldsymbol{\omega}\times] - [(I\delta\boldsymbol{\omega})\times] - \left\{[\boldsymbol{\omega}_r\times]I - [(I\boldsymbol{\omega}_r)\times]\right\}^T\right)I^{-1}\boldsymbol{\lambda}_{\delta\omega}$$

$$\dot{\boldsymbol{\lambda}}_{\delta\zeta} = -\frac{1}{2}Q_3\frac{\partial g}{\partial(\delta\boldsymbol{\zeta})} - \frac{\partial}{\partial(\delta\boldsymbol{\zeta})}\left[\boldsymbol{f}\left(\delta\boldsymbol{\zeta}\right)\delta\boldsymbol{\omega}\right]^T\boldsymbol{\lambda}_{\delta\zeta} - [\boldsymbol{\omega}_r\times]\boldsymbol{\lambda}_{\delta\zeta}$$

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \delta C\boldsymbol{\omega}_r$$

$$\delta\dot{\boldsymbol{\omega}} = -I^{-1}\left\{[(\delta C\boldsymbol{\omega}_r)\times]I - [(I\delta C\boldsymbol{\omega}_r)\times]\right\}\delta\boldsymbol{\omega} - I^{-1}[\delta\boldsymbol{\omega}\times]I\delta\boldsymbol{\omega} + I^{-1}\boldsymbol{u} + [\delta\boldsymbol{\omega}\times]\delta C\boldsymbol{\omega}_r$$
$$-\dot{\boldsymbol{\omega}}_r - I^{-1}[(\delta C\boldsymbol{\omega}_r)\times]I\delta C\boldsymbol{\omega}_r$$

$$\delta\dot{\boldsymbol{\zeta}} = f(\delta\boldsymbol{\zeta})\delta\boldsymbol{\omega}$$

$$\dot{\boldsymbol{\lambda}}_{\delta\omega} = -Q_4\delta\boldsymbol{\omega} - f\left(\delta\boldsymbol{\zeta}\right)^T\boldsymbol{\lambda}_{\delta\zeta} -$$
$$\left(I[\delta\boldsymbol{\omega}\times] - [(I\delta\boldsymbol{\omega})\times] - \left\{[(\delta C\boldsymbol{\omega}_r)\times]I - [(I\delta C\boldsymbol{\omega}_r)\times]\right\}^T + [(\delta C\boldsymbol{\omega}_r)\times]I\right)I^{-1}\boldsymbol{\lambda}_{\delta\omega}$$

$$\dot{\boldsymbol{\lambda}}_{\delta\zeta} = -\frac{1}{2}Q_3\frac{\partial g}{\partial(\delta\boldsymbol{\zeta})} - \frac{\partial}{\partial(\delta\boldsymbol{\zeta})}\left[\boldsymbol{f}\left(\delta\boldsymbol{\zeta}\right)\delta\boldsymbol{\omega}\right]^T\boldsymbol{\lambda}_{\delta\zeta} - \frac{\partial}{\partial(\delta\boldsymbol{\zeta})}\left[\delta\dot{\boldsymbol{\omega}}\right]^T\boldsymbol{\lambda}_{\delta\omega}$$

Optimal Tracking Control

$Q_\omega = Q_\zeta = R = I_{3\times3}$

Open-Loop Optimal Control Problem

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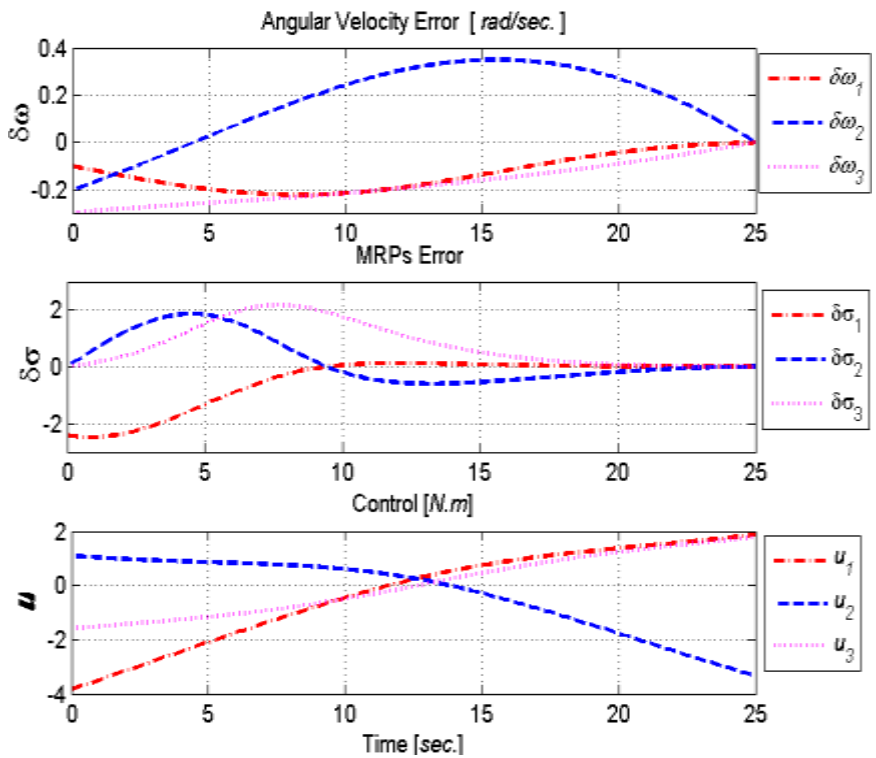
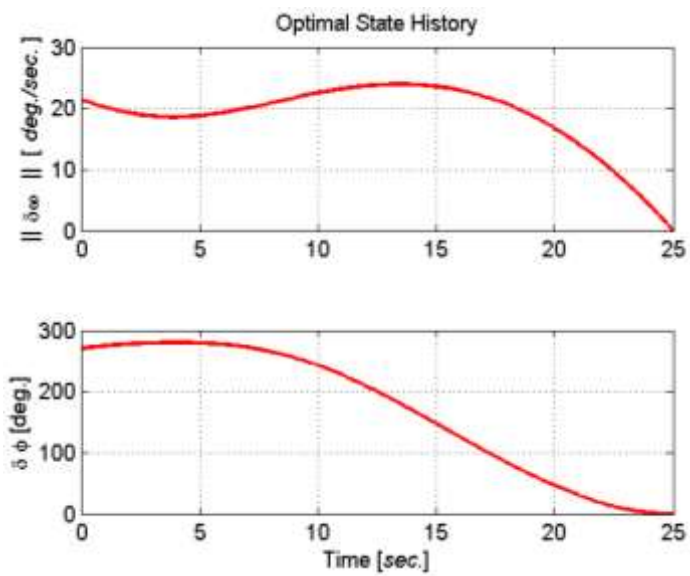




$$\boldsymbol{u} = -R^{-1} I^{-1} \boldsymbol{\lambda}_{\delta\omega} = -(IR)^{-1} \boldsymbol{\lambda}_{\delta\omega}$$

MRP Attitude Penalty Function

Universal Penalty Function

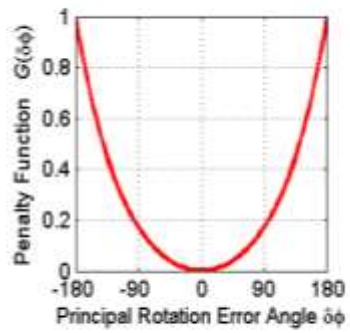
$$G(\delta\boldsymbol{\sigma}) = \delta\boldsymbol{\sigma}^T \delta\boldsymbol{\sigma} = \tan^2(\delta\phi/4)$$

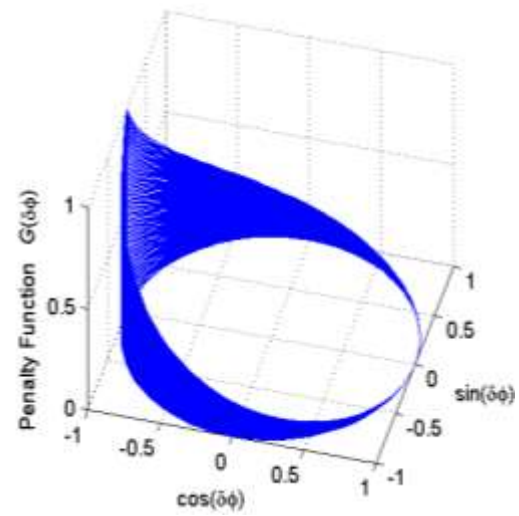
Singularity at $\delta\phi = \pm 2\pi$ is avoided by introducing shadow set

$$\delta\boldsymbol{\sigma}^s = -\delta\boldsymbol{\sigma} / \delta\boldsymbol{\sigma}^T \delta\boldsymbol{\sigma}$$

$$\delta\boldsymbol{\sigma}^s = -\delta\boldsymbol{\sigma}$$

$$\delta\dot{\boldsymbol{\sigma}}^s = \delta\dot{\boldsymbol{\sigma}} - [\delta\boldsymbol{\sigma}\times]\delta\boldsymbol{\omega}$$





The switching surface is set at

$$\delta\boldsymbol{\sigma}^T \delta\boldsymbol{\sigma} = 1$$

$$0 \le |\delta\boldsymbol{\sigma}| \le 1$$

$$0 \le G(\delta\boldsymbol{\sigma}) \le 1$$

Retains the same properties of the function g( )

Conclusion

- Optimal spacecraft maneuver strategies are presented for both open- and closed-loop problem formulations

- Coordinate variable transformations are identified that enable error kinematics models to be developed for arbitrarily large rigid body rotational motion

- Large rotational motion models are presented for several frequently used sets of attitude variables

- Large angular rate dynamics models are presented for supporting closed-loop tracking problem formulations

- Universal penalty functions are introduced to solve for generic optimal tracking control (open-loop and close-loop solutions, large nonlinear motions)

- Numerical simulation results are presented that show the applications of a universal solution to many spacecraft optimal control problems -- remove the dependency on the attitude coordinate choice

Thanks for listening ...!!

Thank You!

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Optimal Open-Loop Control Formulation

Co-states

$$\dot{\boldsymbol{\lambda}}_{\delta\omega} = -Q_4\delta\boldsymbol{\omega} - f\left(\delta\boldsymbol{\zeta}\right)^T \boldsymbol{\lambda}_{\delta\zeta} - \left(I[\delta\boldsymbol{\omega}\times] - [(I\delta\boldsymbol{\omega})\times] - \left\{[\boldsymbol{\omega}_r\times]I - [(I\boldsymbol{\omega}_r)\times]\right\}^T\right) I^{-1}\boldsymbol{\lambda}_{\delta\omega}$$

$$\dot{\boldsymbol{\lambda}}_{\delta\zeta} = -Q_3\delta\boldsymbol{\zeta} - \boxed{\frac{\partial}{\partial(\delta\boldsymbol{\zeta})}\left[\boldsymbol{f}\left(\delta\boldsymbol{\zeta}\right)\delta\boldsymbol{\omega}\right]^T}\boldsymbol{\lambda}_{\delta\zeta} - [\boldsymbol{\omega}_r\times]\boldsymbol{\lambda}_{\delta\zeta}$$

Control

$$\boldsymbol{u} = -R^{-1}I^{-1}\boldsymbol{\lambda}_{\delta\omega} = -(IR)^{-1}\boldsymbol{\lambda}_{\delta\omega}$$

$$\frac{\partial}{\partial(\delta\boldsymbol{\zeta})}\left[\boldsymbol{f}\left(\delta\boldsymbol{\zeta}\right)\delta\boldsymbol{\omega}\right]$$

Quaternion

$$f\left(\delta\boldsymbol{q}\right) = \frac{1}{2}\begin{bmatrix} \delta q_4 & -\delta q_3 & \delta q_2 \\ \delta q_3 & \delta q_4 & -\delta q_1 \\ -\delta q_2 & \delta q_1 & \delta q_4 \\ -\delta q_1 & -\delta q_2 & -\delta q_3 \end{bmatrix} \quad and \quad \frac{\partial}{\partial(\delta\boldsymbol{q})}\left[f\left(\delta\boldsymbol{q}\right)\delta\boldsymbol{\omega}\right] = \frac{1}{2}\begin{bmatrix} -[\delta\boldsymbol{\omega}\times] & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^T & 0 \end{bmatrix}$$

CRPs

$$f\left(\delta\boldsymbol{\rho}\right) = \frac{1}{2}\left[[I_{3\times3}] + [\delta\boldsymbol{\rho}\times] + \delta\boldsymbol{\rho}\delta\boldsymbol{\rho}^T\right] \quad and$$

$$\frac{\partial}{\partial(\delta\boldsymbol{\rho})}\left[f\left(\delta\boldsymbol{\rho}\right)\delta\boldsymbol{\omega}\right] = [I_{3\times3}]\delta\boldsymbol{\rho}^T\delta\boldsymbol{\omega} - \frac{1}{2}[\delta\boldsymbol{\omega}\times]$$

MRPs

$$f\left(\delta\boldsymbol{\sigma}\right) = \frac{1}{4}\left[(1-\delta\boldsymbol{\sigma}^T\delta\boldsymbol{\sigma})[I_{3\times3}] + 2[\delta\boldsymbol{\sigma}\times] + 2\delta\boldsymbol{\sigma}\delta\boldsymbol{\sigma}^T\right] \quad and$$

$$\frac{\partial}{\partial(\delta\boldsymbol{\sigma})}\left[f\left(\delta\boldsymbol{\sigma}\right)\delta\boldsymbol{\omega}\right] = \frac{1}{2}\left[[I_{3\times3}]\delta\boldsymbol{\sigma}^T\delta\boldsymbol{\omega} + \delta\boldsymbol{\sigma}\delta\boldsymbol{\omega}^T - [\delta\boldsymbol{\omega}\times] - \delta\boldsymbol{\omega}\delta\boldsymbol{\sigma}^T\right]$$